Transient heat transfer analysis for moving-boundary transport problems in finite media

V. K. Katiyar
Department of Mathematics, U.O.R., Roorkee, India

B. Mohanty
Department of Chemical Engineering, U.O.R., Roorkee, India

Received 29 January 1987 and accepted for publication 18 November 1987

Problems of heat and mass transfer involving a moving boundary are encountered in freezing, melting, diffusion of species with or without chemical reaction, evaporation, condensation, and in allied fields. In the present investigation the transient solidification of a finite slab of a binary alloy with a moving boundary is considered. The slab is taken to be superheated initially with a uniform temperature distribution. Solidification starts after one surface is cooled by convection while other surfaces are kept insulated. Temperature distribution and the position of moving boundary are determined by solving a non-steady-state energy equation analytically. Predicted results are compared with available experimental results and are found in good agreement.

Keywords: casting; solidification; superheated; mold; mushy

Introduction

Transient heat transfer problems involving melting or freezing are important for many engineering applications, including casting, food processing, polymer production, and welding. Especially in metal casting, where the grain structure of the cast specimen depends on the local time-temperature history of the specimen, transient heat transfer techniques are widely used to predict local and overall phases and associated grain structures.

The above problems are popularly known as moving-boundary problems, due to the movement of different phases with time from one end to the other. Due to the complex nature of solidification phenomena, only its mathematical modeling and subsequent analytical solution become difficult. So far, due to inherent nonlinearities only a few exact solutions for such problems are available in the literature. Neumann[1] has reported an exact solution for the determination of temperature distribution and phase change position of a semi-infinite body. Weiner[2] and Citron[3] have studied identical problems and obtained similar solutions to those of Neumann. Jones[4] has reviewed the application of the approximate solution technique to such problems. In spite of these efforts, the closed forms of exact solutions are rarely available for slabs of finite thickness.

Due to the complex nature of exact and numerical solution techniques available for these problems, the usefulness of a fairly accurate approximate solution looks promising. One such useful technique is the heat balance integral method developed by Goodman.[5] This method has many advanced features. It can be applied to a wide range of problems, and the accuracy obtained is often sufficient for practical purposes. However, it is well known that this method is sensitive to the choice of the approximating temperature profile. It is also difficult to predict the accuracy that may be achieved by a particular profile. Noble[6] suggested the most promising way of systematically improving the accuracy of the heat balance integral method by repeated special subdivision, using quadratic profiles. For each subdivision of the present problem a quadratic temperature profile has been selected.

Statement of the problem

A mold (see Figure 1) of finite dimension was kept in a coolant stream such that cooling of the molten mass took place from the bottom of the mold only. All sides except the bottom were insulated so as to achieve a perfect unidirectional freezing with hardly any convection current in the molten mass. Heat was transferred to the metal wall by conduction and then by convection to the cooling media. The overall heat transfer to coolant for such systems will depend upon the thermophysical properties, velocity of the cooling media, thermal conductivity of the metal wall, and the freezing phase. However, in most cases when cooling is done by air, the air-side heat transfer coefficient controls the overall heat transfer.

For the present work, a one-dimensional solidification of a superheated alloy (50–50 lead–tin) where the phase change occurs over a wide range of temperatures was selected. During freezing, the physical properties of each phase were assumed constant separately.

The whole process of freezing can be divided into five distinct stages, based on the physics of the problem:

Stage 1. Stage 1 begins with the cooling process and ends when the wall temperature drops to T_b—a temperature at which mushy region starts.

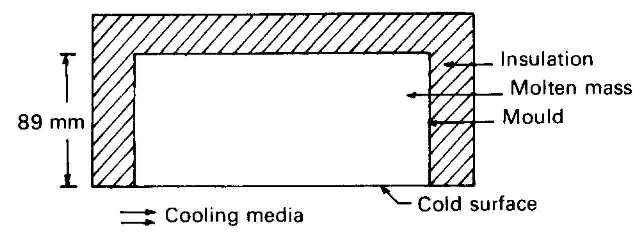

Figure 1 Description of mold

Stage 2. During stage 2 the mushy and liquid regions coexist. Stage 2 ends when the effect of cooling penetrates to the extreme opposite end.

Stage 3. In stage 3 the mushy and liquid regions also coexist. At the end of this stage the complete slab converts the mushy state.

Stage 4. In stage 4 the slab remains in the mushy state. This stage ends when solidification begins at the cooling end of the slab.

Stage 5. The mushy and solid regions coexist in stage 5. This stage ends when the complete slab converts to the solid phase.

Formulation and solution of the problem

In the formulation of the problem, all time periods, basic equations, and boundary conditions have been written in nondimensional form.

First time period

During the first period the temperature at position $X=0$ is reduced to T_b at time t_1. The basic equation and boundary conditions for the affected liquid region in nondimensional form are as follows:

$$\frac{\partial\theta_{11}}{\partial\tau}=\frac{\partial^2\theta_{11}}{\partial X^2}, \quad 0<X<P_1, \quad 0<\tau<\tau_1 \tag{1}$$

$$X=P_1, \quad \theta_{11}=0 \tag{1a}$$

$$X=P_1, \quad \frac{\partial\theta_{11}}{\partial X}=0 \tag{1b}$$

$$X=0, \quad \frac{\partial\theta_{11}}{\partial X}=\frac{1}{R_1^*}\theta_{11} \tag{1c}$$

$$\tau=0, \quad P_1=0 \tag{1d}$$

Using the heat balance integral technique in the affected liquid region along with a cubic temperature profile of the form

$$\theta_{11}=A+BX+CX^2 \tag{2}$$

and boundary conditions 1a–1d, we obtain the value of θ_{11}:

$$\theta_{11}=1-\frac{(P_1-X)^2}{P_1^2+2P_1R_1^*} \tag{3}$$

The first time period ends when the temperature of the free slab surface reaches T_b. The time τ_1 when this occurs is

$$\tau_1=2R_1^{*2}[\theta_{12i}-\ln(1+\theta_{12i})] \tag{4}$$

Second time period

During the second time period, both liquid and mushy regions exist. The mathematical statements of the problem for both regions are as follows:

(i) liquid region

$$\frac{\partial\theta_{12}}{\partial\tau}=\frac{\partial^2\theta_{12}}{\partial X^2}, \quad P_2<X<P_1, \quad \tau_1<\tau<\tau_2 \tag{5}$$

$$X=P_1, \quad \theta_{12}=0 \tag{5a}$$

$$X=P_1, \quad \frac{\partial\theta_{12}}{\partial X}=0 \tag{5b}$$

$$X=P_2, \quad \theta_{12}=\theta_{12b} \tag{5c}$$

$$X=P_2, \quad K_L\frac{\partial\theta_{12}}{\partial X}=\frac{\partial\theta_{22}}{\partial X} \tag{5d}$$

$$\tau=\tau_1, \quad \theta_{12}=\theta_{11} \tag{5e}$$

$$\tau=\tau_1, \quad P_2=0 \tag{5f}$$

Notation

C_p	Specific heat
$\bar{C}_p$	Pseudo specific heat
h	Heat transfer coefficient
k	Thermal conductivity
K_L	Thermal conductivity ratio k_L/k_M
K_M	Thermal conductivity ratio k_M/k_s
L	Slab length
P	Distance of cooling penetration
R_k	Ratio k/h
S	Liquidus phase front position
t_i	Time constant
T_{jk}	Temperature variable
T_0	Ambient temperature
T_i	Initial temperature
T_b	Liquidus temperature
T_v	Solidus temperature
v	Solidus phase front position
X	Distance from cooled surface
α	Thermal diffusivity
$\bar{\alpha}$	Pseudo thermal diffusivity $k(C_p+\bar{C}_p)/\rho$
ρ	Density

Subscripts

k	1, 2, 3, 4, 5 designates time when used with t
j	1, 2, 3 designates liquid, mushy, and solid regions, respectively

Nondimensionalized numbers

θ	$(T_j-T_0)/(T_i-T_0)$
τ	$\alpha_1 t/L^2$
X	x/L
α_{21}	α_2/α_1
α_{31}	α_3/α_1
P_1	P/L
P_2	S/L
P_3	w/L
S_2^*	S_2/L
R_k^*	R_k/L
θ_{12i}	$(T_i-T_b)/(T_b-T_0)$
θ_{12b}	$(T_b-T_0)/(T_i-T_0)$
θ_{120}	$(T_i-T_b)/(T_i-T_0)$
θ_{24v}	$(T_v-T_0)/(T_i-T_0)$
θ_{220}	$T_0/(T_i-T_0)$
θ'_{35v}	$T_0/(T_0-T_v)$
θ_{22b}	$T_b/(T_0-T_b)$
θ'_{220}	$T_0/(T_0-T_b)$
θ_{240}	$T_v/(T_v-T_0)$

(ii) mushy region

$$\frac{\partial\theta_{22}}{\partial\tau}=\frac{\partial^2\theta_{22}}{\partial X^2},\qquad 0<X<P_2,\quad \tau_1<\tau<\tau_2 \tag{6}$$

$$X=P_2,\qquad \theta_{22}=\theta_{12b} \tag{6a}$$

$$X=0,\qquad \frac{\partial\theta_{22}}{\partial X}=\frac{1}{R_2^*}\theta_{22} \tag{6b}$$

The temperature profiles for the liquid and mushy regions can be found in the same way as in the first time period, using the heat balance integral technique.

$$\theta_{12}=1-\theta_{120}\left(\frac{P_1-X}{P_1-P_2}\right)^2 \tag{7}$$

$$\begin{aligned}\theta_{22}=&\left[\theta_{12b}-\frac{K_LP_2}{P_1-P_2}\theta_{12b}+\frac{3P_2}{2R_2^*}\theta_{220}\right]\\&+X\left[\frac{\theta_{12b}}{R_2^*}-\frac{K_LP_2\theta_{12b}}{(P_1-P_2)R_2^*}+\frac{3P_2}{2R_2^{*2}}\theta_{220}-\frac{4\theta_{220}R_2^*}{2R_2^*+3P_2}\right]\\&+X^2\left[\frac{\theta_{220}R_2}{P_2(2R_2^*+3P_2)}-\frac{K_L\theta_{12b}R_2^*}{(P_1-P_2)P_2(2R+3P_2)}-\frac{\theta_{12b}}{2P_2R_2^*}\right.\\&\left.+\frac{K_L\theta_{12b}}{2(P_1-P_2)R_2^*}-\frac{3P_2\theta_{220}}{4P_2R^{*2}}\right]\end{aligned} \tag{8}$$

This period ends when the effect of cooling spans the entire slab; i.e., $P_1=1$, $\tau=\tau_2$. Time is obtained from Equations 6 and 8 using appropriate boundary conditions.

$$\tau_2=\frac{\theta_{22b}}{\frac{2R^*+3P_2}{P_2R_2^{*2}}\left(\theta_{22b}^*\frac{K_LP_2}{1-P_2}+\frac{K_LP_2}{1-P_2}+\frac{3P_2}{2R_2^*}\theta'_{220}\right)+\frac{1}{P_2R_2^*}\theta'_{220}+\frac{K_L}{P_2(1-P_2)}} \tag{9}$$

Third time period

Liquid and mushy regions exist during the third period, which ends when the mushy region reaches the position $X=1$. Basic equations and the boundary conditions for the regions are as follows:

(i) liquid region

$$\frac{\partial\theta_{13}}{\partial\tau}=\frac{\partial^2\theta_{13}}{\partial X^2},\qquad P_2<X<1,\quad \tau_2<\tau<\tau_3 \tag{10}$$

$$X=\frac{\partial\theta_{13}}{\partial X}=0 \tag{10a}$$

$$X=P_2,\qquad \theta_{13}=\theta_{12b} \tag{10b}$$

$$X=P_2,\qquad K_L\frac{\partial\theta_{13}}{\partial X}=\frac{\partial\theta_{23}}{\partial X} \tag{10c}$$

$$\tau=\tau_2,\qquad \theta_{13}=\theta_{12} \tag{10d}$$

$$\tau=\tau_2,\qquad P_2=\frac{S_2}{L}=S_2^* \tag{10e}$$

(ii) mushy region

$$\frac{\partial\theta_{23}}{\partial\tau}=\frac{\partial^2\theta_{23}}{\partial X^2},\qquad 0<X<P_2,\quad \tau_2<\tau<\tau_3 \tag{11}$$

$$X=P_2,\qquad \theta_{23}=\theta_{12b} \tag{11a}$$

$$X=0,\qquad \frac{\partial\theta_{23}}{\partial X}=\frac{1}{R_3^*}\theta_{23} \tag{11b}$$

$$\tau=\tau_2,\qquad \theta_{23}=\theta_{22} \tag{11c}$$

The temperature profile for the liquid region was found by the heat balance integral technique. The mushy region temperature distribution is the same as for the previous period except the time continues to τ_3 and θ_{22} becomes θ_{23} and R_2^* becomes R_3^*.

$$\theta_{13}=\theta_{12b}-\frac{\theta_{12b}}{2(P_1-P_2)(P_2-1)}[2P_2+P_2^2-2X+X^2] \tag{12}$$

$$\begin{aligned}\theta_{23}=&\left[\theta_{12b}-\frac{K_LP_2\theta_{12b}}{P_1-P_2}+\frac{3P_2}{2R_3^*}\theta_{220}\right]\left[1+\frac{X}{R_3^*}-\frac{X^2}{2P_2R_3^*}\right]\\&+\theta_{220}\left(\frac{X^2}{2P_2}-X\right)\frac{2R_3^*}{2R_3^*+3P_2}-\frac{K_L\theta_{12b}}{P_1-P_2}\frac{R_3^*}{(2R_3^*+3P_2)}\frac{X^2}{P_2}\end{aligned} \tag{13}$$

The time τ_3 at which the complete slab is converted into the mushy state is obtained from Equations 11 and 13 and the appropriate boundary conditions.

$$\tau_3=\frac{\theta_{22b}}{\frac{2R_3^*+3P_2}{P_2R_3^{*2}}\left\{(\theta_{22b}+1)\frac{K_LP_2}{1-P_2}+\frac{3P_2}{2R_3^{*2}}\theta'_{220}\right\}+\frac{1}{P_2R_3^*}\theta'_{220}+\frac{K_L}{P_2(1-P_2)}} \tag{14}$$

Fourth time period

Throughout the fourth period, the slab exists in the mushy state and the following equations apply:

$$\frac{\partial\theta_{24}}{\partial\tau}=\frac{\partial^2\theta_{24}}{\partial X^2},\qquad 0<X<1,\quad \tau_3<\tau<\tau_4 \tag{15}$$

$$X=1,\qquad \frac{\partial\theta_{22}}{\partial X}=0 \tag{15a}$$

$$X=0,\qquad \frac{\partial\theta_{24}}{\partial X}=\frac{1}{R_4^*}\theta_{24} \tag{15b}$$

$$\tau=\tau_3,\qquad \theta_{24}=\theta_{23} \tag{15c}$$

Using the heat balance integral technique and applying the initial condition, we obtain the temperature profile for the mushy state:

$$\theta_{24}=\theta_{24v}+\frac{X\theta_{24v}}{R_4^*}+\frac{\theta_{24v}}{2R_4^*}X^2 \tag{16}$$

The period ends at time t_4 when the temperature at position $X=0$ reaches the solidus temperature, which signifies the start of solid formation.

$$\tau_4=\theta_{240}R_4^* \tag{17}$$

Fifth time period

A mushy region and a solid region exist during the fifth period as the solidus phase front moves from position $X=0$ to position $X=1$. The problem formulation for this period becomes the same as the third period if appropriate notational changes are made by replacing the mushy and liquid regions with solid and mushy regions, respectively. The temperature profiles for both regions become

(i) solid region

$$\begin{aligned}\theta_{35}=&\left[\theta_{24v}-\frac{K_MP_2}{P_1-P_3}\theta_{24b}+\frac{3P_3}{2R_5^*}\theta_{220}\right]\left[1+\frac{X}{R_5^*}-\frac{X^2}{2P_3R_5^*}\right]\\&+\theta_{220}\left(\frac{X^2}{2P_3}-X\right)\frac{2R_5^*}{2R_5^*+3P_3}-\frac{K_M\theta_{24b}}{P_1-P_3}\frac{R_5^*}{2R_5^*+3P_3}\frac{X^2}{P_3}\end{aligned} \tag{18}$$

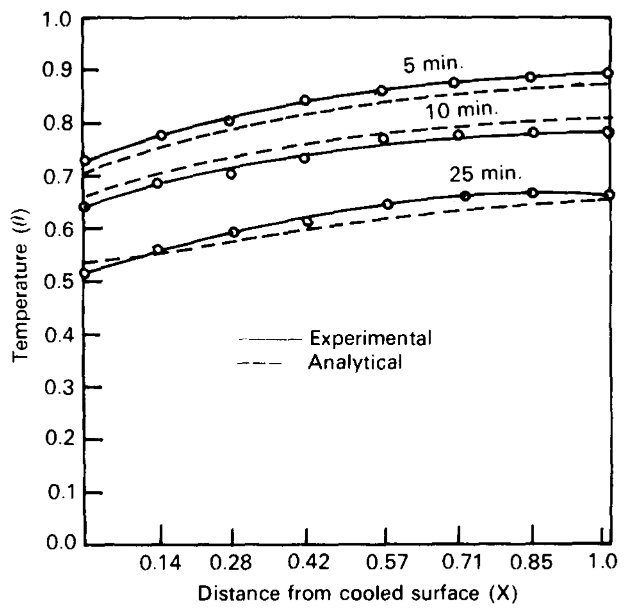

Figure 2 Temperature profiles for solidification of 50–50 tin–lead alloy

(ii) mushy region

$$\theta_{25}=\theta_{24}v-\frac{\theta_{24v}}{2(P_1-P_3)(P_3-1)}[2P_3+P_3^2-2X+X^2] \quad (19)$$

The time period τ_5, when the complete slab converts to the solid phase, has been obtained as in the case of the third time period.

$$\tau_5=\frac{\theta_{35}}{\frac{2R_5^*+3P_3}{P_3R_5^{*2}}\left(\theta_{35v}+\frac{K_MP_3}{1-P_3}+\frac{3P_3}{2R_5^*}\theta'_{35v}\right)+\frac{1}{P_3R_5^*}\theta'_{35v}+\frac{K_M}{P_3(1-P_3)}} \quad (20)$$

Results and discussions

An approximate mathematical model was developed for the freezing of superheated liquid inside a finite mold using the Goodman heat balance integral method. The freezing process was considered to be unidirectional. The process of freezing was considered to pass through five different stages, characterized by the condition of fluid inside the mold and the boundary condition, before it solidifies completely. The necessary equations required to predict the non-steady-state temperature profile along the thickness of the mold and the time period necessary to complete each stage were calculated. To test the extent of prediction of the model, it was tested with the experimental result of Muehlbaure[7] for solidification of a 50–50 lead–tin alloy initially at a superheated temperature of 264°C. The experimental and analytical results are compared in Figure 2. From Figure 2 it is clear that the analytically obtained temperature profile along the thickness of the mold closely agrees with that of experimental value. The maximum deviation observed is less than ±5%.

The deviation in the result can be attributed to many facts, such as uncertainties in the thermophysical properties of a 50–50 lead–tin alloy, because standard property data are not available. The deviation may be partly due to the assumption that physical properties do not vary with temperature for a particular phase.

References

1. Neumann, F. Die Partiellen Differentialgeichungen der Mathematischen. *Physik* 1912, **2**, 121
2. Weiner, J. H. Transition heat conduction in multiphase media. *Br. J. Appl. Phys.* 1955, **6**, 316
3. Citron, J. S. Heat conduction in a melting slab. *J. Aerospace Sci.* 1960, **27**, 219
4. Jones, H. A comparison of approximate analytical solutions of freezing from a plane chill. *J. Inst. Metals* 1969, **97**, 38
5. Goodman, T. R. The heat balance integral and its application to problems involving changes of phase. *Trans. ASME* 1958, **80**, 335
6. Noble, B. *Heat Balance Method in Melting Problems, Moving Boundary Problems in Heat Flow and Diffusion.* Clarendon Press, Oxford, p. 208
7. Muehlbaure, J. C. Transient heat transfer analysis of alloy solidification. *Trans. ASME* 1973, 324